TUTORIAL NOTES FOR MATH4220

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1. FOURIER TRANSFORM AND APPLICATIONS

Let us review some elementary facts about Fourier transform, then applications will be given to study the initial value problem of ordinary differential equations.

1.1. Fourier transform.

Definition 1. The Fourier transform of a function $f : \mathbb{R}^n \to \mathbb{C}$ is defined by

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\xi\cdot x} f(x)dx.$$

Remark 2. Since this short note is not a concise introduction to Fourier transform, in the above definition, we do not specify the well-defined function space for f which ensures the existence of Fourier transform. For the interested readers, we point out Fourier transform is well-defined for the rapidly decreasing function or functions of the so called Schwartz space. Moreover, with the theory of generalized function, Fourier transform can be defined for the slowly increasing function or the so called Tempered distributions which consists of linear functionals on Schwartz space.

Fourier transform has the following properties.

Lemma 3. *Let f, g be complex-valued functions defined on $\mathbb{R}^n$, we have*

(1) $\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$ for all $a, b \in \mathbb{C}$.

(2) For all $y \in \mathbb{R}^n$,

$$\mathcal{F} \circ \tau_y(f) = \mathrm{e}^{-\mathrm{i}y\cdot\xi}\mathcal{F}(f), \quad \mathcal{F}(\mathrm{e}^{\mathrm{i}x\cdot y} f) = \tau_y \circ \mathcal{F}(f),$$

where $\tau_y(f)(x) = f(x - y)$.

(3) $\mathcal{F} \circ \delta_\varepsilon(f) = |\varepsilon|^{-n}\delta_{\varepsilon^{-1}} \circ \mathcal{F}(f)$ for all $0 \neq \varepsilon \in \mathbb{R}$.

(4) $\mathcal{F}(f \circ O) = \mathcal{F}(f) \circ O$, where O is an orthogonal matrix.

(5) $\mathcal{F}(\overline{f}) = \overline{\mathcal{F}(f)}$.

(6) For all $\alpha \in \mathbb{N}^n$,

$$\mathcal{F}(\partial^\alpha f) = (\mathrm{i}\xi)^\alpha \mathcal{F}(f), \quad \partial^\alpha \mathcal{F}(f) = \mathcal{F}[(-\mathrm{i}x)^\alpha f].$$

Lemma 4. *Let f, g be complex-valued functions defined on $\mathbb{R}^n$, we have*

$$\mathcal{F}(f * g) = (2\pi)^{\frac{m}{2}}\mathcal{F}(f) \cdot \mathcal{F}(g), \quad \mathcal{F}(f \cdot g) = (2\pi)^{-\frac{m}{2}}\mathcal{F}(f) * \mathcal{F}(g).$$

Lemma 5 (Parseval's relation)**.** *Let f, g be complex-valued functions defined on $\mathbb{R}^n$, we have*

$$\int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx.$$

Lemma 6. *Let f, g be complex-valued functions defined on $\mathbb{R}^n$, we have*

$$\int_{\mathbb{R}^n} f(\xi)\overline{\hat{g}(\xi)}d\xi = \int_{\mathbb{R}^n} \hat{f}(x)\overline{g(x)}dx.$$

To recover the original function after applying Fourier transform, we have the following result.

Lemma 7. *Let f be a complex-valued function defined on $\mathbb{R}^n$, then*

$$f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\xi\cdot x} \hat{f}(\xi) d\xi.$$

There we give the definition of the inverse Fourier transform.

Definition 8. The inverse Fourier transform of a function $f : \mathbb{R}^n \to \mathbb{C}$ is defined by

$$f^{\vee}(\xi) = \mathcal{F}^{-1}[f](\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\xi\cdot x} f(x) dx.$$

Moreover, we have the following properties between Fourier transform and inver Fourier transform.

Lemma 9. *Let f, g be complex-valued functions defined on $\mathbb{R}^n$, we have*

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)} d\xi = \int_{\mathbb{R}^n} \hat{f}(x)\overline{g^{\vee}(x)} dx.$$

1.2. **Applications.**

1.2.1. *Free fall motion.* Consider the free fall motion of an object in a gravitational and viscous field with the speed u,

$$\begin{aligned} \frac{du}{dt} + \mu u =& g, \quad t \geq 0, \\ u(0) =& v. \end{aligned}$$

Here g is the gravitational constant, μ is the coeffiecient of viscousity. To solve the problem, applying the Fourier transform on both sides of the equation, we have

$$\mathrm{i}\tau\hat{u} + \mu\hat{u} = g\delta,$$

then

$$\hat{u}(\tau) = \frac{g}{\mathrm{i}\tau + \mu}\sqrt{2\pi}\delta(\tau) + \alpha\delta(\tau - \mathrm{i}\mu),$$

where α is a constant to be determined, then we find the general solution to the equation

$$u = \frac{g}{\mu} + \alpha \mathrm{e}^{-\mu t},$$

therefore by the initial condition, the solution to the problem is

$$u(t) = \frac{g}{\mu} + \left(v - \frac{g}{\mu}\right) e^{-\mu t}.$$

1.2.2. *Motion of mass-spring system.* Consider the motion of a mass-spring system,

$$\begin{aligned} \frac{d^2x}{dt^2} + 2\mu\frac{dx}{dt} + k^2 x =& f, \quad t \geq 0, \\ \left(x, \frac{dx}{dt}\right)(0) =& (l, 0). \end{aligned}$$

Here x is the position of a block, $f = a\cos(\omega t)$ is the external force, μ is the coeffiecient of the viscosity, k is torsional constant. To solve the problem, applying the Fourier transform on both sides of the equation, we have

$$-\tau^2\hat{x} + 2\mu\mathrm{i}\tau\hat{x} + k^2\hat{x} = \hat{f},$$

then we split the discussion into two cases.

(1) For $\mu^2 - k^2 = 0$, then $-\tau^2 + \mu\mathrm{i}\tau + k = 0$ has a double root τ_0, therefore

$$\hat{x}(\tau) = \frac{\hat{f}}{-\tau^2 + 2\mu\mathrm{i}\tau + k^2} + \alpha\delta(\tau - \tau_0) + \beta\delta'(\tau - \tau_0),$$

where α, β are constants to be determined, then we find the general solution to the equation

$$x(t) = \frac{a\sin(\omega t)}{2\mu\omega} + \alpha e^{\mathrm{i}\tau_0 t} + \beta\mathrm{i}t\mathrm{e}^{\mathrm{i}\tau_0 t},$$

therefore by the initial condition, the solution to the problem is

$$x(t) = \frac{a\sin(\omega t)}{2\mu\omega} + \frac{\mathrm{e}^{-kt}}{2\mu}[at + 2l(1+kt)\mu]$$

(2) For $\mu^2 - k^2 \neq 0$, then $-\tau^2 + \mu\mathrm{i}\tau + k = 0$ has two distinct roots τ_1, τ_2, therefore

$$\hat{x}(\tau) = \frac{\hat{f}}{-\tau^2 + 2\mu\mathrm{i}\tau + k^2} + \alpha\delta(\tau - \tau_1) + \beta\delta(\tau - \tau_2),$$

where α, β are constants to be determined, then we find the general solution to the equation

$$x(t) = \frac{a(k^2 - \omega^2)\cos(\omega t) - 2a\mu\omega\sin(\omega t)}{(k^2 - \omega^2)^2 + 4\mu^2\omega^2} + \alpha\mathrm{e}^{\mathrm{i}\tau_1 t} + \beta\mathrm{e}^{\mathrm{i}\tau_2 t},$$

therefore by the initial condition, the solution to the problem is

$$\begin{aligned} x(t) =& \frac{a(k^2 - \omega^2)\cos(\omega t) - 2a\mu\omega\sin(\omega t)}{(k^2 - \omega^2)^2 + 4\mu^2\omega^2} \\ &+ \frac{\mathrm{e}^{-t\mu}}{(k^2 - \omega^2)^2 + 4\mu^2\omega^2}[-ak^2 + k^4 l + (a - 2k^2 l + 4l\mu^2)\omega^2 + l\omega^4]\cos(\sqrt{k^2 - \mu^2}t) \\ &+ \frac{\mathrm{e}^{-t\mu}}{\sqrt{k^2 - \omega^2}[(k^2 - \omega^2)^2 + 4\mu^2\omega^2]}[-ak^2 + k^4 l + (3a - 2k^2 l + 4l\mu^2)\omega^2 + l\omega^4]\sin(\sqrt{k^2 - \mu^2}t). \end{aligned}$$

A Supplementary Problem

Problem. Let $\mathcal{F}$ be denoted as

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-\mathrm{i}\xi\cdot x}dx.$$

Prove

$$\mathcal{F}(1) = \delta(\xi), \quad \mathcal{F}(\cos(ax)) = \sqrt{2\pi}\frac{\delta(\xi - a) + \delta(x + a)}{2},$$

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}}e^{-\frac{\xi^2}{4a}}, \quad \mathcal{F}(e^{-a|x|}) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2 + \xi^2},$$

where δ is the Dirac delta function.

For more materials, please refer to [1, 2, 3, 4].

References

[1] S. Alinhac, *Hyperbolic partial differential equations*, Universitext, Springer, Dordrecht, 2009.

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